An Optimal Lower Bound for the Frobenius Problem

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Abstract. Given $N \geq 2$ positive integers $a_1, a_2, \ldots, a_N$ with GCD $(a_1, \ldots, a_N) = 1$, let f_N denote the largest natural number which is not a positive integer combination of $a_1, \ldots, a_N$. This paper gives an optimal lower bound for f_N in terms of the absolute inhomogeneous minimum of the standard $(N-1)$-simplex.

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1 Introduction and statement of results

Given $N \geq 2$ positive integers $a_1, a_2, \ldots, a_N$ with GCD $(a_1, \ldots, a_N) = 1$, the Frobenius problem asks for the largest natural number $g_N = g_N(a_1, \ldots, a_N)$ (called the Frobenius number) such that g_N has no representation as a non–negative integer combination of $a_1, \ldots, a_N$. In this paper, without loss of generality, we assume that $a_1 < a_2 < \ldots < a_N$. The simple statement of the Frobenius problem makes it attractive and the relevant bibliography is very large (see [14] and Problem C7 in [9]). We will mention just few main results.

For $N = 2$, the Frobenius number is given by an explicit formula due to W. J. Curran Sharp [3]:

$$g_2(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1\,.$$

The case $N = 3$ was solved explicitly by Selmer and Beyer [20], using a continued fraction algorithm. Their result was simplified by Rödseth [15] and later by Greenberg [8]. No general formulas are known for $N \geq 4$. Upper bounds, among many others, include classical results by Erdös and Graham [5]

$$g_N \leq 2a_N \left[\frac{a_1}{N}\right] - a_1\,,$$

by Selmer [19]

$$g_N \leq 2a_{N-1} \left[\frac{a_N}{N}\right] - a_N\,,$$

and by Vitek [21]

$$g_N \le \left[\frac{(a_2-1)(a_N-2)}{2}\right] - 1\,,$$

as well as more recent results by Beck, Diaz, and Robins [2]

$$g_N \le \frac{1}{2}\left(\sqrt{a_1a_2a_3(a_1+a_2+a_3)} - a_1 - a_2 - a_3\right)\,,$$

and by Fukshansky and Robins [7], who produced an upper bound in terms of the covering radius of a lattice related to the integers $a_1, \ldots, a_N$.

For $N = 3$, Davison [4] has found a sharp lower bound

$$g_3 \ge \sqrt{3a_1a_2a_3} - a_1 - a_2 - a_3\,,$$

where the constant $\sqrt{3}$ cannot be replaced by any smaller constant. Rödseth [15] proved in the general case that

$$g_N \ge ((N-1)!a_1\cdots a_N)^{1/(N-1)} - \sum_{i=1}^{N} a_i\,.$$

The present paper gives a sharp lower bound for the function

$$f_N(a_1, \ldots, a_N) = g_N(a_1, \ldots, a_N) + \sum_{i=1}^{N} a_i$$

(and thus for g_N) in terms of a geometric characteristics of the standard $(N-1)$-simplex. Clearly, $f_N = f_N(a_1, \ldots, a_N)$ is the largest integer which is not a *positive* integer combination of $a_1, \ldots, a_N$.

Following the geometric approach developed in [12] and [13], we will make use of tools from the geometry of numbers. Recall that a family of sets in $\mathbb{R}^{N-1}$ is a *covering* if their union equals $\mathbb{R}^{N-1}$. Given a set S and a lattice L, we say that L is a *covering lattice* for S if the family $\{S+\mathbf{l} : \mathbf{l} \in L\}$ is a covering. Recall also that the *inhomogeneous minimum* of the set S with respect to the lattice L is the quantity

$$\mu(S, L) = \inf\{\sigma > 0 : L \text{ a covering lattice of } \sigma S\}$$

and the quantity

$$\mu_0(S) = \inf\{\mu(S, L) : \det L = 1\}$$

is called the *absolute inhomogeneous minimum* of S. If S is bounded and has inner points, then $\mu_0(S)$ does not vanish and is finite (see [11], Chapter 3).

Let S_{N-1} be the standard simplex given by

$$S_{N-1} = \{(x_1, \ldots, x_{N-1}) : x_i \geq 0 \,\text{reals and}\, \sum_{i=1}^{N-1} x_i \leq 1\}\,.$$

The main result of the paper shows that the constant $\mu_0(S_{N-1})$ is a sharp lower bound for (suitably normalized) Frobenius number and integers with relatively small f_N are, roughly speaking, dense in $\mathbb{R}^{N-1}$.

Theorem 1.1. (i) *For $N \geq 3$ the inequality*

$$\mu_0(S_{N-1}) \leq \frac{f_N(a_1, \ldots, a_N)}{(a_1 \cdots a_N)^{1/(N-1)}} \tag{1}$$

holds.

(ii) *For any $\epsilon > 0$ and for any point $\alpha = (\alpha_1, \ldots, \alpha_{N-1})$ in $\mathbb{R}^{N-1}$ there exist N integers $0 < a_1 < a_2 < \ldots < a_N$ with* $\mathrm{GCD}\,(a_1, \ldots, a_N) = 1$ *such that*

$$\left| \alpha_i - \frac{a_i}{a_N} \right| < \epsilon\,, \quad i = 1, 2, \ldots, N-1 \tag{2}$$

and

$$\frac{f_N(a_1, \ldots, a_N)}{(a_1 \cdots a_N)^{1/(N-1)}} < \mu_0(S_{N-1}) + \epsilon\,. \tag{3}$$

Remark 1.1. Prof. L. Davison kindly informed the authors that the part (i) of Theorem 1.1 was proved by Rödseth in [16] without using geometry of numbers.

The quantity $\mu_0(S)$ is closely related to the *covering constant* $\Gamma(S)$ of the set S, where

$$\Gamma(S) = \sup\{\det(L) : L \text{ a covering lattice of } S\}\,. \tag{4}$$

By Theorem 1, Ch. 3, §21 of [11] (see also [1]) for each Lebesgue measurable set S

$$\Gamma(S) \leq \mathrm{vol}\,(S)\,, \tag{5}$$

and by Theorem 2 ibid.

$$\mu_0(S) = \frac{1}{\Gamma(S)^{1/(N-1)}}\,. \tag{6}$$

The proof of Theorem 1, Ch. 3, §21 of [11] easily implies that the equality in (5) is attended only if S is a space–filler. Further, by Theorem 6.3 of [17], packings of simplices cannot be very dense and, consequently, S_{N-1} is not a space–filler. Therefore, by (5) and (6),

$$\mu_0(S_{N-1}) > \frac{1}{(\mathrm{vol}\,(S_{N-1}))^{1/(N-1)}} = ((N-1)!)^{1/(N-1)}\,, \tag{7}$$

and we get the following result.

Corollary 1.1. *For $N \geq 3$ the inequality*

$$f_N(a_1, \ldots, a_N) > ((N-1)!a_1 \cdots a_N)^{1/(N-1)} \tag{8}$$

holds.

The inequality (8) with non–strict sign was proved in [16]. The only known value of $\mu_0(S_{N-1})$ is $\mu_0(S_2) = \sqrt{3}$ (see e. g. [6]). In the latter case we get the following slight generalization of Theorems 2.2 and 2.3 in [4].

Corollary 1.2. *For $N = 3$ the inequality*

$$f_3(a_1, a_2, a_3) \geq (3a_1a_2a_3)^{1/2}$$

holds. Moreover, for any $\epsilon > 0$ and for any point $\alpha = (\alpha_1, \alpha_2)$ in $\mathbb{R}^2$ there exist integers $0 < a_1 < a_2 < a_3$ with $\mathrm{GCD}\,(a_1, a_2, a_3) = 1$ such that

$$\left| \alpha_i - \frac{a_i}{a_3} \right| < \epsilon\,, \quad i = 1, 2$$

and

$$f_3(a_1, a_2, a_3) < ((3+\epsilon)a_1a_2a_3)^{1/2}\,.$$

Let us consider a lattice M in $\mathbb{R}^{N-1}$ generated by the vectors

$$\frac{1}{N-1}\mathbf{e}_1, \ldots, \frac{1}{N-1}\mathbf{e}_{N-1}\,, \tag{9}$$

where $\mathbf{e}_j$ are the standard basis vectors. Since the fundamental cell of M w. r. t. the basis (9) belongs to S_{N-1}, the lattice M is a covering lattice for the simplex S_{N-1}. Therefore, by (4) and (6),

$$\mu_0(S_{N-1}) \leq \frac{1}{(\det M)^{1/(N-1)}} = N - 1\,.$$

This implies the following result.

Corollary 1.3. *For any $\epsilon > 0$ and for any point $\alpha = (\alpha_1, \ldots, \alpha_{N-1})$ in $\mathbb{R}^{N-1}$ there exist N integers $0 < a_1 < a_2 < \ldots < a_N$ with* $\mathrm{GCD}\,(a_1, \ldots, a_N) = 1$ *such that*

$$\left| \alpha_i - \frac{a_i}{a_N} \right| < \epsilon \,, \quad i = 1, 2, \ldots, N-1$$

and

$$\frac{f_N(a_1, \ldots, a_N)}{(a_1 \cdots a_N)^{1/(N-1)}} < N - 1 + \epsilon \,.$$

Remark 1.2. Note that the inequality (7) and Stirling's formula imply that

$$\liminf_{N\to\infty} \frac{\mu_0(S_{N-1})}{N-1} \geq e^{-1} \,.$$

Thus, we know the asymptotic behavior of the optimal constant $\mu_0(S_{N-1})$ up to the multiple e.

For $\mathbf{a} = (a_1, a_2, \ldots, a_N)$, define a lattice $L_{\mathbf{a}}$ by

$$L_{\mathbf{a}} = \{(x_1, \ldots, x_{N-1}) : x_i \text{ integers and } \sum_{i=1}^{N-1} a_i x_i \equiv 0 \mod a_N\} \,.$$

The following theorem is implicit in [18].

Theorem 1.2. *For any lattice L with basis $\mathbf{b}_1, \ldots, \mathbf{b}_{N-1}$, $\mathbf{b}_i \in \mathbb{Q}^{N-1}$, $i = 1, \ldots, N-1$ and for all rationals $\alpha_1, \ldots, \alpha_{N-1}$ with $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{N-1} \leq 1$, there exists an infinite arithmetic progression $\mathcal{P}$ and a sequence*

$$\mathbf{a}(t) = (a_1(t), \ldots, a_{N-1}(t), a_N(t)) \in \mathbb{Z}^N \,, t \in \mathcal{P} \,,$$

such that $\mathrm{GCD}\,(a_1(t), \ldots, a_{N-1}(t), a_N(t)) = 1$ *and the lattice $L_{\mathbf{a}(t)}$ has a basis*

$$\mathbf{b}_1(t), \ldots, \mathbf{b}_{N-1}(t)$$

with

$$\frac{b_{ij}(t)}{d\,t} = b_{ij} + O\left(\frac{1}{t}\right) \,, \quad i, j = 1, \ldots, N-1 \,, \tag{10}$$

where $d \in \mathbb{N}$ is such that $d\, b_{ij}, d\, \alpha_j\, b_{ij} \in \mathbb{Z}$ for all $i, j = 1, \ldots, N-1$. Moreover,

$$a_N(t) = \det(L) d^{N-1} t^{N-1} + O(t^{N-2}) \tag{11}$$

and

$$\alpha_i(t) := \frac{a_i(t)}{a_N(t)} = \alpha_i + O\left(\frac{1}{t}\right) . \tag{12}$$

For completeness, we give a proof of Theorem 1.2 in Section 4.

2 Proof of Theorem 1.1 (i)

Recall that $\mathbf{a} = (a_1, a_2, \ldots, a_N)$ and put

$$\alpha_1 = \frac{a_1}{a_N}, \ldots, \alpha_{N-1} = \frac{a_{N-1}}{a_N}.$$

Define a simplex $S_{\mathbf{a}}$ by

$$S_{\mathbf{a}} = \{(x_1, \ldots, x_{N-1}) : x_i \geq 0 \text{ reals and } \sum_{i=1}^{N-1} a_i x_i \leq 1\}.$$

Theorem 2.5 of [12] states that

$$f_N(a_1, \ldots, a_N) = \mu(S_{\mathbf{a}}, L_{\mathbf{a}}). \tag{13}$$

Observe that the inhomogeneous minimum $\mu(S, L)$ satisfies

$$\mu(S, tL) = t\mu(S, L),$$

$$\mu(tS, L) = t^{-1}\mu(S, L).$$

Thus, if we define

$$S_\alpha = a_N S_{\mathbf{a}} = \{(x_1, \ldots, x_{N-1}) : x_i \geq 0 \text{ reals and } \sum_{i=1}^{N-1} \alpha_i x_i \leq 1\},$$

$$L_u = a_N^{-1/(N-1)} L_{\mathbf{a}}$$

then

$$\mu(S_{\mathbf{a}}, L_{\mathbf{a}}) = a_N^{1+1/(N-1)} \mu(S_\alpha, L_u). \tag{14}$$

Note that $\det L_{\mathbf{a}} = a_N$. Thus the lattice L_u has determinant 1 and we have

$$\mu_0(S_\alpha) \leq \mu(S_\alpha, L_u). \tag{15}$$

The simplices $(\alpha_1 \cdots \alpha_{N-1})^{1/(N-1)} S_\alpha$ and S_{N-1} are equivalent up to a linear transformation of determinant 1. Therefore

$$\mu_0(S_{N-1}) = \frac{\mu_0(S_\alpha)}{(\alpha_1 \cdots \alpha_{N-1})^{1/(N-1)}}, \tag{16}$$

and by (15), (14) and (13) we have

$$\mu_0(S_{N-1}) \leq \frac{\mu(S_\alpha, L_u)}{(\alpha_1 \cdots \alpha_{N-1})^{1/(N-1)}} = \frac{\mu(S_{\mathbf{a}}, L_{\mathbf{a}})}{a_N^{1+1/(N-1)} (\alpha_1 \cdots \alpha_{N-1})^{1/(N-1)}}$$

$$= \frac{f_N(a_1, \ldots, a_N)}{(a_1 \cdots a_N)^{1/(N-1)}}.$$

3 Proof of Theorem 1.1 (ii)

The proof is based on Theorem 1.2 and the following continuity property of the inhomogeneous minima. We say that a sequence S_t of *star bodies* in $\mathbb{R}^{N-1}$ converges to a star body S if the sequence of *distance functions* of S_t converges uniformly on the unit ball in $\mathbb{R}^{N-1}$ to the distance function of S.

Lemma 3.1. *Let S_t be a sequence of star bodies in $\mathbb{R}^{N-1}$ which converges to a bounded star body S and let L_t be a sequence of lattices in $\mathbb{R}^{N-1}$ convergent to a lattice L. Then*

$$\lim_{t\to\infty} \mu(S_t, L_t) = \mu(S, L)\,.$$

Proof. The result follows from a much more general Satz 1 of [10]. □

W. l. o. g., we may assume that $\alpha \in \mathbb{Q}^{N-1}$ and

$$0 < \alpha_1 < \alpha_2 < \ldots < \alpha_{N-1} < 1\,. \tag{17}$$

For $\epsilon > 0$ we can choose a lattice L_ϵ of determinant 1 with

$$\mu(S_\alpha, L_\epsilon) < \mu_0(S_\alpha) + \frac{\epsilon(\alpha_1 \cdots \alpha_{N-1})^{1/(N-1)}}{2}\,. \tag{18}$$

The inhomogeneous minimum is independent of translation and rational lattices are dense in the space of all lattices. Thus, by Lemma 3.1, we may assume that $L_\epsilon \subset \mathbb{Q}^{N-1}$. Applying Theorem 1.2 to the lattice L_ϵ and the numbers $\alpha_1, \ldots, \alpha_{N-1}$, we get a sequence $\mathbf{a}(t)$, satisfying (10), (11) and (12). Note also that, by (17),

$$0 < a_1(t) < a_2(t) < \ldots < a_N(t)$$

for sufficiently large t.

Observe that the inequality (12) implies (2) with $a_i = a_i(t)$, $i = 1, \ldots, N$, for t large enough. Let us show that, for sufficiently large t, the inequality (3) also holds. Define a simplex $S_{\alpha(t)}$ and a lattice L_t by

$$S_{\alpha(t)} = a_N(t) S_{\mathbf{a}(t)} = \{(x_1, \ldots, x_{N-1}) : x_i \geq 0 \,\text{reals and}\, \sum_{i=1}^{N-1} \alpha_i(t) x_i \leq 1\}\,,$$

$$L_t = a_N(t)^{-1/(N-1)} L_{\mathbf{a}(t)}\,.$$

By (10) and (11), the sequence L_t converges to the lattice L_ϵ. Next, the point $\mathbf{p} = (1/(2N), \ldots, 1/(2N))$ is an inner point of the simplex S_α and all the simplicies $S_{\alpha(t)}$ for sufficiently large t. By (12) and Lemma 3.1, the sequence $\mu(S_{\alpha(t)} - \mathbf{p}, L_t)$

converges to $\mu(S_\alpha - \mathbf{p}, L_\epsilon)$. Since the inhomogeneous minimum is independent of translation, the sequence $\mu(S_{\alpha(t)}, L_t)$ converges to $\mu(S_\alpha, L_\epsilon)$. Consequently, by (12),

$$\frac{\mu(S_{\alpha(t)}, L_t)}{(\alpha_1(t)\cdots\alpha_{N-1}(t))^{1/(N-1)}} \to \frac{\mu(S_\alpha, L_\epsilon)}{(\alpha_1\cdots\alpha_{N-1})^{1/(N-1)}}\,, \quad \text{as } t\to\infty\,,$$

and, by (13), (18) and (16),

$$\frac{f_N(a_1(t),\ldots,a_N(t))}{(a_1(t)\cdots a_N(t))^{1/(N-1)}} = \frac{\mu(S_{\alpha(t)}, L_t)}{(\alpha_1(t)\cdots\alpha_{N-1}(t))^{1/(N-1)}} < \mu_0(S_{N-1}) + \epsilon$$

for sufficiently large t.

4 Proof of Theorem 1.2

Let us consider the matrices

$$B = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1\,N-1} & \sum_{i=1}^{N-1}\alpha_i b_{1i} \\ b_{21} & b_{22} & \ldots & b_{2\,N-1} & \sum_{i=1}^{N-1}\alpha_i b_{2i} \\ \vdots & \vdots & & \vdots & \vdots \\ b_{N-1\,1} & b_{N-1\,2} & \ldots & b_{N-1\,N-1} & \sum_{i=1}^{N-1}\alpha_i b_{N-1\,i} \end{pmatrix}$$

and

$$M = M(t, t_1, \ldots, t_{N-1})$$

$$= \begin{pmatrix} db_{11}t + t_1 & db_{12}t & \ldots & db_{1\,N-1}t & d\sum_{i=1}^{N-1}\alpha_i b_{1i}t \\ db_{21}t & db_{22}t + t_2 & \ldots & db_{2\,N-1}t & d\sum_{i=1}^{N-1}\alpha_i b_{2i}t \\ \vdots & \vdots & & \vdots & \vdots \\ db_{N-1\,1}t & db_{N-1\,2}t & \ldots & db_{N-1\,N-1}t + t_{N-1} & d\sum_{i=1}^{N-1}\alpha_i b_{N-1\,i}t \end{pmatrix}.$$

Denote by $M_i = M_i(t, t_1, \ldots, t_{N-1})$ and B_i the minors obtained by omitting the ith column in M or in B, respectively. Following the proof of Theorem 2 in [18], we observe that

$$|B_N| = |\det(b_{ij})| = \det L\,, \tag{19}$$

$$|B_i| = \alpha_i |B_N|\,, \tag{20}$$

$$M_i = d^{N-1}B_i t^{N-1} + \text{polynomial of degree less than } N-1 \text{ in } t\,, \tag{21}$$

and $M_1, \ldots, M_N$ have no non–constant common factor.

By Theorem 1 of [18] applied with $m = 1$, $F = 1$, and $F_{1\nu} = M_\nu(t, t_1, \ldots, t_{N-1})$, $\nu = 1, \ldots, N$, there exist integers $t_1^*, \ldots, t_{N-1}^*$ and an infinite arithmetic progression $\mathcal{P}$ such that for $t \in \mathcal{P}$

$$\mathrm{GCD}(M_1(t, t_1^*, \ldots, t_{N-1}^*), \ldots, M_N(t, t_1^*, \ldots, t_{N-1}^*)) = 1\,.$$

Put

$$\mathbf{a}(t) = (M_1(t, t_1^*, \ldots, t_{N-1}^*), \ldots, (-1)^{N-1} M_N(t, t_1^*, \ldots, t_{N-1}^*))\,, \quad t \in \mathcal{P}\,.$$

Then the basis $\mathbf{b}_1(t), \ldots, \mathbf{b}_{N-1}(t)$ for $L_{\mathbf{a}(t)}$ satisfying the statement of Theorem 1.2 is given by the rows of the matrix obtained by omitting the Nth column in the matrix $M(t, t_1^*, \ldots, t_{N-1}^*)$. The properties (19)–(21) of minors M_i, B_i imply the properties (10)–(12) of the sequence $\mathbf{a}(t)$, $t \in \mathcal{P}$.

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